# Effective-field theories for the $\nu=\frac{5}{2}$ quantum Hall edge state

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The quantum Hall state at the filling fraction $\nu=5/2$ is the leading candidate to be a physical system supporting excitations with non-Abelian braid statistics. While a direct experimental evidence for the latter is still absent, recent experiments by Radu *et al.* [I. P. Radu, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, Science **320**, 899 (2008)] yielded results favoring some previously proposed non-Abelian theories over Abelian ones. Here, we systematically investigate candidate theories of the quantum Hall edge at the filling fraction $\nu=5/2$. We find a set of candidate theories, both Abelian and non-Abelian, which are equally consistent with the experimental data and cannot therefore be distinguished in the quasihole tunneling experiment only. We discuss what experimental information may be useful in resolving the ambiguity.

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It was pointed out long ago[1] that in a (2+1)-dimensional universe there might exist quantum particles, anyons, obeying neither Fermi nor Bose statistics. For all its elegance this result did not have any obvious relations to the nature, where fundamental particles are fermions and bosons. The discovery of the fractional quantum Hall effect revolutionized this view—it was found that anyons may emerge as excitations (quasiparticles) above a strongly correlated state of a many-electron system. Further on, Wen[2] and Moore and Read[3] brought forward arguments attempting to explain the $\nu=5/2$ conductance plateau as a manifestation of a strongly correlated state supporting excitations with non-Abelian braid statistics. Not only is this possibility fascinating from the theoretical point of view, it also holds promise for concrete implementations of topologically protected quantum algorithms.[4] Today, the $\nu=5/2$ quantum hall edge (QHE) state remains the most promising candidate to be the non-Abelian state.

Notwithstanding persisting interest of the community to the problem, the crucial properties of the $\nu=5/2$ state do, however, remain hypothetical. There is no controllable microscopic theory of the system due to the complexity of the Hamiltonian of electrons in a quantizing magnetic field. At the same time exact numerical studies do not allow one to make fully reliable statements about macroscopically large systems. The decisive word about the nature of the $\nu=5/2$ state must therefore come from experiments. A natural target for experimental investigations is the edge of an incompressible quantum Hall fluid supporting gapless excitations.

In a recent experiment,[5] properties of the $\nu=5/2$ state were investigated by means of a transport measurement in a quantum Hall sample with a narrow constriction. The parameters of the constriction were tuned in such a way that it served as a weak link between two $\nu=5/2$ quantum Hall edges. The electrical conductance of the constriction exhibits a zero-bias peak, whose scaling with temperature is consistent with the assumption that the current is due to weak tunneling of fractionally charged quasiparticles. By fitting the shape of the zero-bias conductance peak to the predictions of a model-independent theory[6] at five different temperatures, the experimentalists produced a two-parameter confidence map for the electric charge $e^*$ and the scaling dimension $g$ of the tunneling quasiparticle.

In order for these data to be useful in unveiling the nature of the $\nu=5/2$ state, it is crucial to the values of $e^*$ and $g$ in various possible effective theories. In this Brief Report, rather than relying on prejudices based on aesthetic, microscopic, or numerical arguments to sift out candidate theories, we present a list of theories satisfying a certain minimal set of physical assumptions. We show that, although the knowledge (within experimental errors) of the two parameters of Ref. 5 narrows down the list of candidate effective theories significantly, it is impossible to reveal their non-Abelian nature based *exclusively* on these tunneling data. Based on our analysis, we discuss which additional experiments may help to achieve this goal and test the crucial predictions of the theory of QHE.

*General principles.* We first recall some general principles underlying the effective-field theory of the quantum Hall edge (for details, see Refs. 7–9) and then focus on their application to the $\nu=5/2$ state. We assume that the effective theory of a QHE is a chiral conformal field theory (CFT) that meets the following requirements imposed by fundamental properties of the system.

(A) The CFT at the edge supports a *chiral* current $J$ that is not conserved due to the inflow of a Hall current from the incompressible bulk. It is convenient to use a chiral Bose field $\phi$ related to the charge density $J^0$ by

$$J^0(x) = \frac{\sqrt{\nu}}{2\pi}\partial_x\phi(x), \tag{1}$$

where $x$ is a natural parameter along the edge and $\phi$ satisfies the commutation relations

$$[\phi(x),\phi(x')] = i\pi\,\mathrm{sign}(x-x'). \tag{2}$$

The factor $\sqrt{\nu}$ in Eq. (1) is dictated by the electric charge conservation (anomaly cancellation) in the system.

(B) Since microscopically the system is composed of electrons, the chiral CFT must contain a local operator $\psi_e$ of unit charge representing the electron in the effective theory

$$[J_0(x),\psi_e(y)] = -\delta(x-y)\psi_e(y). \tag{3}$$

 

TABLE I. The charges $e^*$ and the scaling dimensions $g$ of the most relevant tunneling operators in $N=2,3$ theories. The parameters of the $K$ matrices defined in Eq. (8) are shown by the symbols $\binom{b}{a}$ for $N=2$ and $\binom{a_1,a_2;b}{l_1,l_2,l_3}$, for $N=3$.

| $K$ | $\binom{1}{3}$ | $\binom{-1}{5}$ | $\binom{2,1;2}{3,3,3}$ | $\binom{1,2;2}{3,3,5}$ | $\binom{1,1;1}{3,5,5}$ | $\binom{4,0;-1}{5,5,5}$ | $\binom{3,1;-1}{5,5,5}$ | $\binom{2,2;-1}{5,5,5}$ |
|---|---|---|---|---|---|---|---|---|
| $e^*$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $g$ | $\frac{3}{8}$ | $\frac{5}{24}$ | $\frac{1}{2}$ | $\frac{11}{24}$ | $\frac{7}{32}$ | $\frac{9}{40}$ | $\frac{1}{4}$ | $\frac{7}{24}$ |

(C) Fundamentally, any correlation function of the theory containing an electron-operator insertion must be a single-valued function of the position of the insertion. In the effective-theory language this means that $\psi_e$ must be *local* with respect to all primary fields of the CFT.

A physically plausible effective theory should satisfy certain minimality conditions. Complicated theories with very rich spectra of quasiparticles, large central charge, and large scaling dimension of electron operators may be unstable, e.g., against the formation of a Wigner crystal.[10]

In connection with the $\nu=5/2$ state, it is usually assumed that the cyclotron gap is quite large and the electrons from the filled lowest Landau level (with one spin-up and one spin-down electron per orbital) do not participate in the formation of the strongly correlated state. Electrons in the *half-filled* second Landau level form a strongly correlated $\nu=1/2$ incompressible state which we study here. (An alternative picture for the case of nonchiral states has been considered in Ref. 11).

Equation (3) implies that the electron operator in the effective theory may be written as

$$\psi_e(x) = e^{i\sqrt{2}\phi(x)}W(x), \tag{4}$$

where $W(x)$ describes neutral degrees of freedom. Note that neither $W(x)$ nor $e^{i\sqrt{2}\phi}$ must be local fields in the field content of the effective CFT. By Eqs. (1)–(3) it is seen that $\psi_e(x)$ has unit charge. The need for additional degrees of freedom described by $W(x)$ becomes clear if one considers the permutation relation $e^{i\sqrt{2}\phi(x)}e^{i\sqrt{2}\phi(y)}=e^{-i\theta}e^{i\sqrt{2}\phi(y)}e^{i\sqrt{2}\phi(x)}$, where the statistical parameter $\theta=2\pi$ [see Eq. (2)]. It follows that for $W=1$ the operator $\psi_e$ possesses Bose statistics and hence cannot describe an electron. Our goal is then to describe the neutral degrees of freedom of the edge. We call a chiral CFT "Abelian" or "non-Abelian" depending on whether its primary fields obey Abelian or non-Abelian statistics, respectively. In this Brief Report we limit our analysis to two simple cases: (1) chiral Abelian theories and (2) chiral CFTs where the neutral sector is decoupled from the charged one and $\psi_e(x)=e^{i\sqrt{2}\phi(x)}\otimes W(x)$, where $W(x)$ is a primary field in a non-Abelian CFT. In both cases we shall see that there exist plausible theories that are in much better agreement with the experiment than, e.g., the Pfaffian state.

*Abelian theories.* Chiral Abelian edge CFTs are constructed from a multiplet $\phi=(\phi_1,\dots,\phi_N)$ of free chiral bosons satisfying $[\phi_i(x),\phi_j(x')]=i\pi\delta_{ij}\,\mathrm{sgn}(x-x')$ that give rise to $N$ conserved currents $J_i^\mu=(2\pi)^{-1}\epsilon^{\mu\nu}\partial_\nu\phi_i$. The electric current is a linear combination $J_{\rm el}=\mathbf{q}\cdot\mathbf{J}\equiv\Sigma_i q_iJ_i$, where $q_i$ are some coefficients. A general excitation is described by a vertex operator

$$\psi_{\mathbf{v}} = e^{i\mathbf{v}\cdot\boldsymbol{\phi}} \tag{5}$$

with the statistical parameter $\theta=\pi\mathbf{v}\cdot\mathbf{v}$ and the charge $Q_{\rm el}=\mathbf{q}\cdot\mathbf{v}$. If operator (5) represents an electron, $\theta$ must be odd and $Q_{\rm el}=1$. Imposing this condition, we find $N$ solutions $\mathbf{v}=\mathbf{e}_\alpha$, $\alpha=1,\dots,N$. The theory is characterized by its $K$ matrix $K_{\alpha\mu}=\mathbf{e}_\alpha\cdot\mathbf{e}_\mu$, whose entries are mutual statistical phases of electron operators. Using anomaly cancellation condition $\mathbf{q}\cdot\mathbf{q}=\nu$, one finds

$$\mathbf{Q}K^{-1}\mathbf{Q} = \nu, \quad \mathbf{Q} = (1,1,\dots,1). \tag{6}$$

Condition (C) implies that an arbitrary excitation (5) satisfies $\mathbf{v}\cdot\mathbf{e}_\alpha=n_\alpha$, where $n_\alpha\in\mathbb{Z}$. The conformal spin and the electric charge of such an excitation are given by

$$h(\mathbf{n}) = \mathbf{n}K^{-1}\mathbf{n}, \quad Q_{\rm el}(\mathbf{n}) = \mathbf{Q}K^{-1}\mathbf{n}. \tag{7}$$

Equations (6) and (7) can be used for a complete classification of Abelian $\nu=1/2$ states. As an illustration, we discuss $N=2$ Abelian states and state the results for $N=3$. In Ref. 10 it has been shown that, for physically interesting states with small relative angular momentum of electrons, $K$ matrices can always be chosen in the form

$$K = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad K = \begin{pmatrix} l_1 & a_1 & a_2 \\ a_1 & l_2 & b \\ a_2 & b & l_3 \end{pmatrix}, \tag{8}$$

where $l_1\le l_2\le l_3$. For $N=2$ fields and $\nu=1/2$ Eq. (6) reduces to $a+b=4$, where $a\in 2\mathbb{Z}+1$. In such a theory the electric charge is given by $Q_{\rm em}=(n_1+n_2)/(a+b)=(n_1+n_2)/4$. There are only three $K$ matrices (with det $K>0$, for the theory to be chiral) for which the conformal spin of an electron $h_e\le 7/2$.

A complete classification of irreducible $N=3$ theories can be found in Ref. 10. There are six distinct indecomposable three-dimensional chiral lattices describing admissible $\nu=1/2$ QHE states. The parameters of the $K$ matrices, the charges, and the scaling dimensions of the most relevant tunneling operators in all these theories [calculated using Eq. (7)] are given in Table I. Comparing with experiment [see Fig. 1(a)], one can see that there are two good candidate theories, both with $N=3$. One of them has a minimal charge $e^*=1/8$.

*Non-Abelian theories.* A generalization of the above construction may be obtained if one replaces the Heisenberg algebra describing the modes of free bosons with a central extension of some Lie algebra $\mathfrak{g}$. The resulting theory contains several non-Abelian currents $J_a$ satisfying the Kac-Moody (KM) algebra $\hat{\mathfrak{g}}_k$,

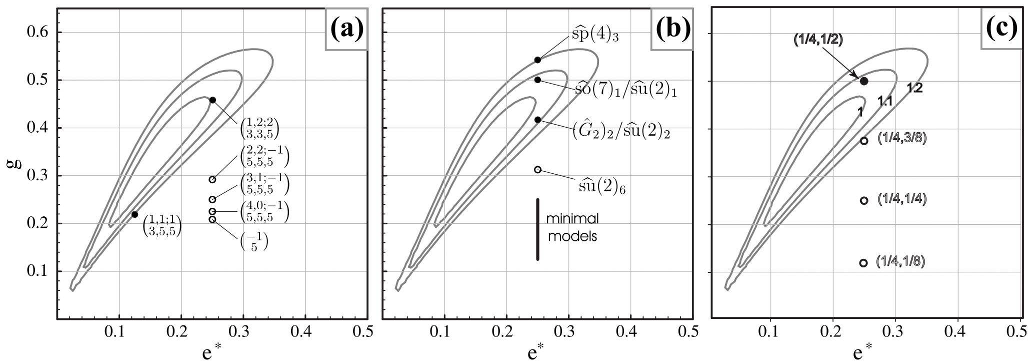


FIG. 1. Predictions of (a) Abelian and (b) non-Abelian theories discussed in the present work are shown on the $(e^{*},g)$ plane and compared with the (c) fit quality map from Ref. 5. Theories lying within the $3\sigma$ contour are shown as full circles. Points corresponding to previously discussed proposals are shown in (c). In particular, the point (1/4,1/2) in (c) corresponds to $\widehat{\mathrm{su}}(2)_2$ and the anti-Pfaffian states and (1/4,1/4) to the Moore-Read state.

$$[J_a(x),J_b(0)] = 2\pi i k\delta_{ab}\delta'(x) + 2\pi i\delta(x)\sum_c f_{ab}^c J_c(0),$$

where $f_{ab}^c$ are the structure constants of $\mathfrak{g}$ and $k$ is a parameter called the *level*. We do not attempt to explore all such theories. Instead, we focus our attention on theories where (a) the electric current commutes with other KM currents; (b) the electron operator is given by Eq. (4), where $W$ is a KM primary field; and (c) $\mathfrak{g}$ is a simple Lie algebra. In the following we call the corresponding CFT the *neutral sector*. All non-Abelian $\nu=5/2$ states proposed so far are of this type. Unitary CFTs associated with Lie algebras are Wess-Zumino-Witten (WZW) models or coset theories generated from WZW models by means of the so-called Goddard-Kent-Olive construction (see, e.g., Ref. 12). Among them are theories based on KM algebras at level 1, which may give rise to Abelian CFTs (see Refs. 10 and 13 for applications to the QHE).[14]

The requirement that the electron operator has Fermi statistics imposes $W(x)W(y)=-W(y)W(x)$, i.e., the neutral sector must contain a primary field of half-integer conformal spin. Not every such a field is, however, acceptable. Indeed, the operator product expansion of a pair of primary fields in the neutral sector is generally given by

$$\phi_a(z)\phi_b(0) = \sum_c C_{ab}^c \phi_c z^{h_c-h_a-h_b} + \text{descendants}, \quad (9)$$

where $h_i$ is the conformal dimension of the field $\phi_i$. In general, the dimensions $h_i$ are not commensurate. Substituting $W$ instead of $\phi_a$ in Eq. (9) one can see that, in order to satisfy the locality requirement (C), the right-hand side of Eq. (9) must contain exactly one primary field for every $\phi_b$. In the language of fusion rules this is expressed as $W\times\phi_b=\phi_c$, i.e., fusion with $W$ determines a permutation on the set of primary fields. Such a primary field $W$ is called a *simple current*.[15] The presence of a half-integer-spin simple current in the theory is a strong constraint.

Quasihole excitations are described by operators of the neutral and the charged sectors as

$$\psi_{\mathrm{qh}} = e^{iq\sqrt{2}\phi(x)} \otimes V(x), \quad (10)$$

where $q$ is the quasihole charge and $V(x)$ is a Virasoro primary field in the neutral sector. Substituting Eqs. (4) and (10) in Eq. (9) and imposing the locality constraint (C), one finds that the conformal weights satisfy

$$h_{W\times V} - h_W - h_V + 2q \in \mathbb{Z}. \quad (11)$$

An important property of spectrum of electric charges of the edge CFT is expressed in terms of the *order* of the simple current $W$, defined[15] as the smallest integer $\ell$ such that $W^{\ell}=\mathbb{I}$. It is shown[13] that

$$q \in (\ell d_H)^{-1}\mathbb{Z}, \quad (12)$$

where $d_H$ is the *Hall denominator*, $d_H=2$ in our case. The dimension of the tunneling operator $\psi_{\mathrm{qh}}^{\dagger}\psi_{\mathrm{qh}}$ is

$$g = 2(h_V + q^2). \quad (13)$$

*Wess-Zumino-Witten models.* One of the early proposals for a non-Abelian neutral sector is the $\widehat{\mathrm{su}}(2)_2$ theory.[2] This theory has a central charge $c=3/2$ and contains an SU(2) triplet of Majorana-Weyl fermions with $h_W=1/2$ (order $\ell=2$ simple currents) and a doublet of quasihole excitations with $h_{\mathrm{qh}}=3/16$. In this theory $W$ is identified with the Majorana-Weyl triplet. The charge of the most relevant quasihole is $e^{*}=1/4$ and the dimension of the tunneling operator is $g=2(3/16+1/16)=1/2$. Among the theories discussed in Ref. 5 this model fits the experiment data best.

To generalize the $\widehat{\mathrm{su}}(2)_2$ theory, one may consider $\widehat{\mathrm{su}}(2)$ at higher levels or different Lie algebras. With increasing level $k$ or rank $r$ of $\mathfrak{g}$, the complexity of the CFT increases, while the smallest dimension $h_V$ decreases. From these observations one can deduce that only a limited number of WZW models are plausible candidates compatible with experiment data. This allows one to obtain all plausible WZW theories on a case by case basis.

The interesting candidate theories obtained as a result of this analysis are described in Table II. The smallest fractional charge $e^{*}$ in all these theories is 1/4 Comparison with experiment is shown in Fig. 1(c). The model based on $\widehat{\mathrm{sp}}(4)_1$ is the simplest one and is similar to $\widehat{\mathrm{su}}(2)_2$ containing one multiplet

TABLE II. The central charge $c$, the number $P$ of KM primary fields, the conformal spin $h_e$ of the electron operator, and the tunneling dimension $g$ for non-Abelian models ($\nu=1/2$).

| Model | $\widehat{\mathrm{su}}(2)_6$ | $\widehat{\mathrm{sp}}(4)_1$ | $\widehat{\mathrm{sp}}(4)_3$ | $\widehat{\mathrm{so}}(7)_1/\widehat{\mathrm{su}}(2)_1$ | $(\widehat{G}_2)_2/\widehat{\mathrm{su}}(2)_2$ |
|---|---|---|---|---|---|
| $c$ | 9/4 | 5/2 | 5 | 5/2 | 19/6 |
| $P$ | 7 | 3 | 10 | 6 | 12 |
| $h_e$ | 5/2 | 3/2 | 5/2 | 3/2 | 3/2 |
| $g$ | 5/16 | 3/4 | 13/24 | 1/2 | 5/12 |

of electrons and one of quasiholes. It, however, predicts too large a value of $g$. The best fit to the experiment corresponds to $\widehat{\mathrm{sp}}(4)_3$.

*Coset models.* A much bigger class of conformal field theories (all known rational unitary CFTs) is obtained via the GKO coset construction. For example, the Moore-Read state[3] is associated with the coset $\widehat{\mathrm{su}}(2)_1 \oplus \widehat{\mathrm{su}}(2)_1/\widehat{\mathrm{su}}(2)_2$ describing the chiral sector of the critical Ising model. An exhaustive analysis of the possibilities offered by the coset construction is the subject of future work. Here, we discuss some simple examples.

(a) *Virasoro minimal models.* These form an infinite series of CFTs associated with the cosets $\widehat{\mathrm{su}}(2)_k \oplus \widehat{\mathrm{su}}(2)_1/\widehat{\mathrm{su}}(2)_{k+1}$. The properties of these models, which exhaust all unitary CFTs with central charge $c<1$, are described in the literature in great detail. For $k=4m-3$ and $k=4m-2$, where $m \in \mathbb{Z}$ the model contains an order 2 fermion simple current, identified with $W$. Its conformal spin grows rapidly with $m$ and becomes 15/2 already for $m=2$, so that theories with $m>2$ are implausible from the point of view of stability. For all models in the series one finds the minimal charge $e^*=1/4$ and the dimension $g$ of the most relevant tunneling operator is monotonically decreasing from $g=1/4$ to $g=1/8$ [solid line in Fig. 1(b)]. Thus, in this series the Moore-Read state gives the best fit to the experiment and is preferred from the point of view of stability.

(b) Another interesting class of coset models with $c \geq 1$ are *super-Virasoro minimal models.* Due to supersymmetry a simple current with a conformal weight of 3/2 is present in the spectrum. In this series $e^*=1/4$. The dimensions $g$ of the most relevant tunneling operators lie between $g=59/280$ and $g=1/8$ [solid line in Fig. 1(b)].

(c) As examples of more general coset models we mention here $\widehat{\mathrm{so}}(7)_1/\widehat{\mathrm{su}}(2)_1$ and $(\widehat{G}_2)_2/\widehat{\mathrm{su}}(2)_2$ [see Fig. 1(b) and Table II].

In conclusion, combining the fundamental requirements (A)–(C) with bounds imposed by the experiment[5] we have shown that there is a limited number of chiral conformal field theories that may serve as effective-field theories for the $\nu=5/2$ edge [see Figs. 1(b) and 1(c)]. Some of these theories have not been previously discussed. Intriguingly, there exist both Abelian and non-Abelian states with exactly the same values for $e^*$ and $g$ as the Pfaffian, anti-Pfaffian, and $\widehat{\mathrm{su}}(2)_2$ states discussed in Ref. 5. Thus, it is impossible to distinguish between these states and, in particular, reveal their non-Abelian nature based on the tunneling data of Ref. 5 *only*. The ambiguity might be resolved with the help of further experimental data. In particular, information on the tunneling density of states of electrons[16] would set bounds on the maximal value of the conformal spin of the electron, noise measurements[17] might be used to verify the theory predicting a minimal charge of 1/8, Aharonov-Bohm interferometry might be used to detect conformal dimensions of different excitations,[18] and the multiplicity of electron operator could be measured in the setup of Ref. 19.

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